expression for the short nozzle admittance that is frequency independent. Substituting this expression for the nozzle admittance into the relationship derived by Cantrell and Hart,[5] the theoretical formula $\Lambda_N = -\bar{M}\ (\gamma+1)/2$ is obtained, where γ is the specific heat ratio and $\bar{M}$ is the mean flow Mach number in the chamber. Letting $\gamma = 1.4$, the theoretically predicted value of Λ_N for the test nozzle has been calculated, and it also is presented in Fig. 2. A comparison of the predicted and measured values of Λ_N shows that the predicted data are lower than the corresponding measured values and that the data obtained from the standing-wave method are in best agreement with the theoretical predictions.

It is believed that the main reason for the observed discrepancy between the theoretical and experimental results is that the cross-sectional areas of the combustor and the test nozzle are not equal in the experimental setup (see Fig. 1 for details). Consequently, the flow stream separates from the simulated chamber walls at some location upstream of the nozzle entrance plane which results in a nonuniform steady flow region just upstream of the nozzle entrance plane. This region may result in additional attenuation that is reflected in the measured values of Λ_N. The losses associated with the nonuniform flow region are not accounted for in the theoretical analysis of Ref. 6, and hence the predicted value of Λ_N accounts for damping provided by the nozzle only.

It obviously is desirable to have the capability for separating the acoustic losses associated with the nozzle and the region of nonuniform flow just upstream of the nozzle entrance. One way to obtain such data is to compare the data reported in this note with decay data obtained when the tested nozzle smoothly connects into the simulated combustor. As no such tests were conducted during this investigation, the aforementioned comparison cannot be carried out. However, useful observations can be made by considering the data reported in Ref. 2. In this study, the admittances of a family of liquid rocket nozzles whose entrance areas equaled the cross-sectional area of the simulated combustor have been measured and compared with the theoretical predictions of the Crocco's nozzle admittance theory.[7] An examination of the data presented in Ref. 2 (i.e., see Figs. 4-6 of Ref. 2) indicates a good agreement between the theoretically predicted and experimentally measured data.

The predictions of Crocco's nozzle admittance theory asymptotically approach those of the quasisteady short nozzle theory in the limit when the length of the convergent section of the nozzle is much smaller than the wavelength of the oscillation. Hence, the observed agreement[2] between the measured and predicted nozzle admittance data throughout the frequency range strongly suggests that the short nozzle theory also can be expected to predict the acoustic characteristics of short nozzles that smoothly connect with the combustor. This discussion also suggests that the observed discrepancy between the measured and predicted data can be due to the manner in which the nozzle is attached to the combustor. This further indicates that caution must be exercised whenever available theories are used to predict the admittances and decay data of nozzles having complex geometries in the vicinity of the nozzle entrance plane.

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Numerical Solution for Subcritical Flows by a Transonic Integral Equation Method

Wandera Ogana*
NASA Ames Research Center, Moffett Field, Calif.

Introduction

THE integral equation for two-dimensional transonic flows[1-3] involves a singular kernel. Numerical solution of the equation has been attempted by various methods.[2-7] A major computational challenge concerns the proper treatment of the singular kernel so as to represent accurately the contribution from the double integral. In some schemes,[2,3,5] the double integral is reduced to a line integral over the airfoil by representing the transverse variation of the velocity in terms of the velocity on the airfoil surface, using an arbitrary approximation function that attenuates away from the airfoil. Solutions are obtained only on the airfoil. Radbill[4] and Nixon[6,7] extend the idea of approximation functions to enable computation of flow field values.

For the method presented in this article, the region of integration is divided into closed rectangular elements, and the velocity is taken to be constant in each element. The integral equation is then reduced to a matrix equation solvable by some suitable iteration scheme.

Derivation and Solution of the Matrix Equation

Let $\bar{\delta}$ be the thickness ratio of a thin airfoil, in the physical rectangular coordinates $(\bar{x},\bar{y})$, whose upper profile is described by $\bar{y} = \bar{Y}_+(\bar{x})$ and lower profile by $\bar{y} = \bar{Y}_-(\bar{x})$. For a freestream Mach number, $M_\infty < 1$, and a transonic similarity parameter, $K = (1-M_\infty^2)/M_\infty^{4/3}\bar{\delta}^{2/3}$, the coordinate transformations, $x = \bar{x}$ and $y = (1-M_\infty^2)^{1/2}\bar{y}$, enable the integral equation to be written in the form[1]

$$u(x,y) = u_B(x,y) + \nu u^2(x,y) + \int\!\!\int_S \kappa(\xi - x, \zeta - y)\, u^2(\xi,\zeta)\, dS \tag{1}$$

where

$$\kappa(\xi - x, \zeta - y) = -\frac{(\xi - x)^2 - (\zeta - y)^2}{4\pi[(\xi - x)^2 + (\zeta - y)^2]^2} \tag{2a}$$

$$\nu = (1/\pi)\arctan(\lambda) \tag{2b}$$

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*NRC Research Associate.

where λ is the aspect ratio of the vanishing rectangular cavity that surrounds the singularity.[1]

$$u_B(x,y)=\frac{1}{2\pi}\int_0^1\left\{\frac{2y(\xi-x)}{[(\xi-x)^2+y^2]^2}\Delta\phi(\xi)\right.$$

$$\left.-\frac{\xi-x}{[(\xi-x)^2+y^2]}\Delta\phi_\zeta(\xi)\right\}d\xi \qquad (3)$$

where

$$\Delta\phi(\xi)=\phi(\xi,+0)-\phi(\xi,-0) \qquad (4a)$$

$$\Delta\phi_\zeta(\xi)=\frac{\gamma+1}{\delta K^{3/2}}\left(\frac{d\bar{Y}_+}{d\bar{\xi}}-\frac{d\bar{Y}_-}{d\bar{\xi}}\right) \qquad (4b)$$

The quantity $\phi(\xi,\zeta)$ is related to the perturbation velocity potential $\bar{\phi}(\bar{\xi},\bar{\zeta})$ by

$$\phi(\xi,\zeta)=[(\gamma+1)M_\infty^2/(1-M_\infty^2)]\bar{\phi}(\bar{\xi},\bar{\zeta}) \qquad (5a)$$

and $u(x,y)$ is related to the streamwise perturbation velocity $\bar{u}(\bar{x},\bar{y})$ by

$$u(x,y)=[(\gamma+1)M_\infty^2/(1-M_\infty^2)]\bar{u}(\bar{x},\bar{y}) \qquad (5b)$$

The integral equation is reduced to a form more convenient for the development of a numerical scheme by using the following notations[8]

$$P=(x,y),\quad Q=(\xi,\zeta),\quad dS_P=dxdy,$$

$$dS_Q=d\xi d\zeta,\quad \kappa(\xi-x,\zeta-y)=\kappa(Q,P) \qquad (6)$$

Equation (1) becomes

$$u(P)=u_B(P)+\nu u^2(P)+\int_S\kappa(Q,P)u^2(Q)dS_Q \qquad (7)$$

The region of integration is now divided into n rectangular elements Δ_1, Δ_2, ..., Δ_n, containing the mesh points $P_1(x_1,y_1)$, $P_2(x_2,y_2)$,..., $P_n(x_n,y_n)$. Each mesh point is located as shown in Fig. 1. For each mesh point P_i, Eq. (7) becomes

$$u(P_i)=u_B(P_i)+\nu u^2(P_i)+\sum_{j=1}^{n}\int_{\Delta_j}\kappa(Q,P_i)u^2(Q)dS_Q \qquad (8)$$

It is now assumed that the velocity $u(Q)$ is constant within each element Δ_j and can be represented by its value at the mesh point P_j, that is

$$u(Q)=u(P_j),\quad \text{for } Q \text{ in } \Delta_j \qquad (9)$$

Let

$$u_i\equiv u(P_i),\quad u_{Bi}\equiv u_B(P_i),\quad u_j^2\equiv u^2(P_j) \qquad (10)$$

Define the matrix $A=[a_{ij}]$ by

$$a_{ij}=\nu\delta_{ij}+\int_{\Delta_j}\kappa(Q,P_i)dS_Q,\quad i,j=1,2,...,n \qquad (11)$$

Equation (8) becomes

$$u_i=u_{Bi}+\sum_{j=1}^{n}a_{ij}u_j^2,\qquad i=1,2,...,n \qquad (12)$$

Define the component $g_i(u)$ of a vector $g(u)$ by

$$g_i(u)=u_{Bi}+\sum_{j=1}^{n}a_{ij}u_j^2,\quad i=1,2,...,n \qquad (13)$$

Equation (12) becomes

$$u=g(u) \qquad (14)$$

where u and $g(u)$ are defined by

$$u=[u_1,u_2,...,u_n]^T \qquad (15)$$

$$g=[g_1(u),g_2(u),...,g_n(u)]^T \qquad (16)$$

The matrix A is nonsingular.[1] This guarantees the solution of Eq. (14) by an iteration scheme.[9] For the Jacobi iteration, the initial value is taken to be

$$u^{(0)}=u_B \qquad (17a)$$

and successive iterates are defined by

$$u^{(m+1)}=g[u^{(m)}],\qquad m=0,1,2,... \qquad (17b)$$

where the superscript (m) means the mth iterate.

Although more powerful methods exist for solving Eq. (14), it is found that the Jacobi iteration also leads to fast convergence.

Matrix Elements

Written out in full, Eq. (11) is

$$a_{ij}=\nu\delta_{ij}-\frac{1}{4\pi}\int_{\Delta_j}\int\frac{(\xi-x_i)^2-(\zeta-y_i)^2}{[(\xi-x_i)^2+(\zeta-y_i)^2]^2}d\xi d\zeta$$

$$i,j=1,2,...,n \qquad (18)$$

If $P_i(x_i,y_i)$ is not located in Δ_j, off-diagonal elements are determined by evaluating Eq. (18) at once. Diagonal elements are obtained when $\Delta_j\equiv\Delta_i$, making the integral in Eq. (18) singular at $P_i(x_i,y_i)$ so that, before integration is performed, the singularity is enclosed in a vanishing rectangular cavity of aspect ratio λ (Fig. 1). By substituting the limits of integration in Eq. (18) it can be established that[1]

$$a_{ii}=(1/2\pi)[\arctan(r_i/\delta_i)+\arctan(q_i/\delta_i)] \qquad (19)$$

$$a_{ij}=(1/4\pi)[\arctan(N/D)],\qquad i\neq j \qquad (20)$$

where δ_i, r_i, and q_i are defined as shown in Fig. 1 and

$$N=2\delta_j(r_j+q_j)[(y_j-y_i+r_j)(y_j-y_i-q_j)$$

$$-(x_j-x_i+\delta_j)(x_j-x_i-\delta_j)] \qquad (21a)$$

$$D=[(x_j-x_i-\delta_j)^2+(y_j-y_i+r_j)(y_j\text{-}y_i\text{-}q_j)]$$

$$\times[(x_j-x_i+\delta_j{}^2+(y_j\text{-}y_i+r_j)(y_j\text{-}y_i\text{-}q_j)]$$

$$+(r_j+q_j)^2(x_j-x_i+\delta_j)(x_j\text{-}x_i-\delta_i) \qquad (21b)$$

If $r_j+q_j=2h_j$ is the height of Δ_j then the quantities q_j and r_j are chosen as follows

$$q_j=r_j=h_j,\quad P_j \text{ not on the } \xi \text{ axis} \qquad (22a)$$

$$q_j=0 \text{ and } r_j=2h_j,\quad P_j \text{ on the } \xi \text{ axis} \qquad (22b)$$

Depending on how the region of integration is partitioned, the matrix A displays some interesting symmetries which have been examined by the author.[1]

Results

If the airfoil profile is defined by a simple function, the quantity u_{Bi} in Eq. (12) can be obtained by straightforward

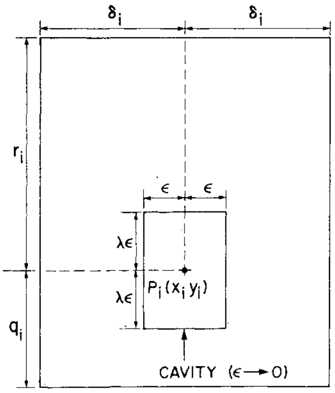

Fig. 1 Location of mesh point in rectangular element.

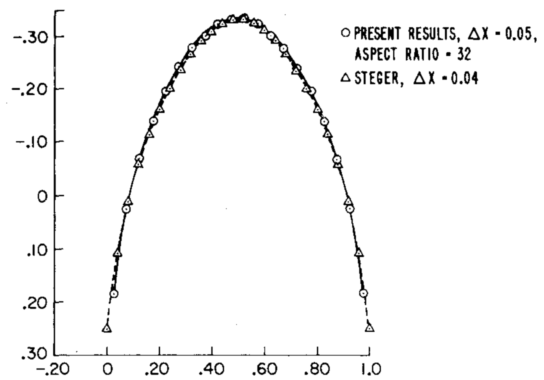

Fig. 2 Coefficient of pressure for a parabolic-arc airfoil of thickness ratio 0.06, at $M_\infty = 0.825$.

integration of Eq. (3). In most cases, however, the integral has to be evaluated numerically. Radbill[4] describes a numerical integration scheme suitable for an arbitrary sharp-edged airfoil.

The ideas described in this article were implemented for a nonlifting parabolic-arc airfoil of thickness ratio $\bar{\delta} = 0.06$. Discussion of the results is best done through aspect ratios of the elements (Fig. 1).

$$\text{aspect ratio of } \Delta_j = (\text{height of } \Delta_j)/(\text{width of } \Delta_j)$$

$$= (r_j + q_j)/2\delta_j \tag{23}$$

Figure 2 shows the coefficient of pressure plots at freestream Mach number 0.825, for the present method, using 280 high-aspect ratio elements, and for a finite-difference method by J. L. Steger of NASA Ames Research Center, using a network of 150×64 mesh points. In the transformed coordinates, the region of integration extended about one chord length from the airfoil edges, in the streamwise direction, and eight chord lengths in the transverse direction.

Convergence was attained in 11 iterations for $M_\infty = 0.825$ and in less than 12 iterations for all subcritical Mach numbers tested. Most of the computing time was used in fulfilling the convergence criterion for mesh points near the airfoil. Consequently, the time per iteration was substantially decreased by not computing $g[u^{(m)}]$ in Eq. (17b) for any mesh point where the convergence criterion was already satisfied. The effect on the coefficient of pressure was insignificant.

For supercritical flows up to $M_\infty = 0.87$, symmetrical results were yielded. Extension of the method to calculation of supercritical flows with shocks has been made very recently and will be reported in a future publication.

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Technical Comments

Comment on "Flutter of a Panel Supported on an Elastic Foundation"

Alexander H. Flax*
Institute for Defense Analyses, Arlington, Va.

CHOPRA[1] has given relationships between theoretical frequency-damping coefficient values on the flutter boundary for a panel supported on an elastic foundation in terms of the values for the same panel unsupported. For supersonic panel flutter, when aerodynamic forces based on linearized quasistatic or piston theory are assumed, such relationships are inherent in the parameters of the theoretical analysis of the flutter problem as given by numerous authors, including Movchan[2] and Dugundji[3] (Ref. 6 of Chopra's Note). Unfortunately, Chopra's statement of the relationships is in error and he has also apparently assumed that they apply to more general theories for the aerodynamic forces and to subsonic panel flutter, which is not the case.

On the flutter boundary, where the variation of the motion in time is purely sinusoidal, the appropriate relationships between the flutter frequencies ω_F and damping coefficients g_T (which, as defined by Dugundji, is a viscous damping coefficient and not the so-called structural damping coefficient usually denoted by g) are, in the notation used by Dugundji and Chopra

$$\omega_{Fe}^2 = \omega_F^2 + K/m \tag{1}$$

and

$$\omega_{Fe} g_{Te} = \omega_F g_T \tag{2}$$

where K is the stiffness of the foundation per unit area of panel, m is the mass per unit area of the panel, and the subscript e refers to the panel with elastic support.

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*President. Fellow AIAA.